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A Remark on the Definition of Superselection Rules in Terms of Unbounded Operators

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Abstract

The mathematical definition of superselection rules in the case when observables are described by unbounded operators in a fixed Hilbert space (for instance, in the frame of Wightman's axioms) is examined. The additional condition $P_{H_q}D \subset D$ (where D is the common domain of definition of the operators, H_q is the qth sector, and P_{H_q} is the projection on H_q) is found to be sufficient in order to preserve—as in the case of bounded observables—the one-to-one correspondence between reducing subspaces H_q and projections P_{H_q} from the commutant $\mathscr{A}'$ of the algebra $\mathscr{A}$ of observables. This additional condition is equivalent to the physical requirement that every physical vector state can be uniquely represented as a linear combination of physical states, each belonging to some sector.

In the following Note, observations are made on some elementary facts from reduction theory of sets of unbounded operators in Hilbert space H. These facts are used to define mathematically superselection rules for the case when the observables of the quantum system are described directly in terms of unbounded operators but are not substituted by bounded operators. The description is similar to the one given by Streater & Wightman (1964). To achieve a complete analogy with the case of bounded operators, the introduction of an additional condition in the definition of superselection sectors is found to be necessary. (It is automatically fulfilled in the case of bounded observables.) If $D(D \neq H)$ is the common domain of definition of the unbounded operators used as observables and H_q is the subspace of H representing the qth sector, the condition $P_{H_q}D \subset D$ should be fulfilled with P_{H_q} the projection on H_q. This turns out to be equivalent to the physical requirement that every physical vector state from D can be uniquely represented as a linear combination of physical states from D, each belonging to some sector H_q. In the course of the discussion an interesting result of

Powers (1971) on reducing subspaces for algebras of unbounded operators acquires its natural generalization.

We add some remarks justifying our interest in the use of unbounded observables. Mathematically, the observables could be treated in two ways: as self-adjoint linear operators, or more generally as "operations"—as used by Haag & Kastler (1964)—which are represented in the general case by unbounded linear, but not necessarily self-adjoint, operators. (Then the additional condition is found to be necessary if one describes mathematically unbounded observables in both ways.) If a quantum field satisfying Wightman's axioms is given, the set of observables can be determined, for instance as the gauge-invariant part of the field in the sense of Doplicher et al. (1969) or by some other explicit construction. The observables represented by unbounded operators should be defined on the common dense domain D in the hilbert space H on which the field is defined also. If we think of observables as self-adjoint operators, it is sometimes convenient to substitute them by their spectral resolutions, as was done by Streater & Wightman (1964) and Jauch & Misra (1961), in discussing superselection rules. An argument in favor of the direct use of unbounded observables is given by Streater & Wightman (1964) and is based on the equations of motion which contain unbounded operators. Another simple argument in the same direction is the following: If one transforms A, the unbounded observables, by a function φ having the inverse φ^{-1} so that $\varphi(A)$ is bounded, there is a one-to-one correspondence between the measured values of A and $\varphi(A)$, but the algebraic and some other relations between observables are not conserved.

Denote by $\mathscr{A} \equiv \{A\}$ the set of all observables. Linear operators $\{A\}$ (among which there are unbounded ones) are defined together with their adjoints $\{A^*\}$ on the dense linear domain D (the domain of physical states) in H so that $AD \subset D, A^*D \subset D$, i.e., D is invariant for $\mathscr{A}$. (We note that if we restrict A on D, $\mathscr{A}$ becomes an algebra.) Given a concrete operator algebra $\mathscr{A}$ of observables, the occurrence of superselection rules is expressed mathematically by the reduction of the set $\mathscr{A}$ (but not of the field) so that $H(\mathscr{A})$ is a direct sum $H = \oplus_q H_q$ ($\mathscr{A} = \oplus \mathscr{A}_q$) and the sectors H_q are "invariant" subspaces for $\mathscr{A}$. In the case where $\mathscr{A}^q$ consists of bounded operators only, the reduction of $\mathscr{A}$ is simply described by the commutant $\mathscr{A}'$ of $\mathscr{A}$, see, e.g., Naimark (1968). If $\mathscr{A}$ contains unbounded operators also, it may happen that $\mathscr{A}'$ consists only of operators multiple of the unity operator, but there exist in H subspaces which are invariant for $\mathscr{A}$. Such an example is discussed by Powers (1971) for the case of a Schrödinger representation of commutation relations. To avoid such "anomalies" we use as physically interesting only a restricted class of invariant subspaces. The closed linear subspace H_1 we call invariant for $\mathscr{A}$ if together with the property $A(H_1 \cap D) \subset H_1$ for $A \in \mathscr{A}$ the additional condition

$$P_{H_1} D \subset D \tag{1}$$

is fulfilled, where P_{H_1} is the projection on H_1. With this definition, the projection of every physical state from D is again a physical state. In the sequel we demonstrate the following proposition: if H_1 is an invariant subspace for $\mathscr{A}$, then its orthogonal complement $H_1{}^\perp$ is also an invariant subspace for $\mathscr{A}$. From the additional condition (1) it follows that any state from D (for instance a mixed state) can be represented uniquely as a finite linear combination of physical states from D contained in the individual sectors H_q. Conversly, if for every $f \in D$ we have $f = \sum_q d_q f_q$, with $f_q \in H_q \cap D$, then using $f_q \equiv P_{H_q} f$ we thus verify the validity of the condition (1). It follows that condition (1) is equivalent to the above-stated physical requirement.

If $\mathscr{A}$ consists only of self-adjoint operators (see Akhieser & Glasmann, 1966, n. 46, Theorem 5), or if $\mathscr{A}$ is an algebra of bounded operators with involution (see Naimark, 1968), Sec. 17, 1), the proposition is true–i.e., each invariant subspace is automatically a reducing subspace (its orthoganal complement is also invariant).

To demonstrate the proposition we use a modification of Naimark's proof (Sec. 17, 1). The modification is necessary because of the inequality $D \neq H$. Note first if $P_{H_1} D \subset D$ then also $P_{H_1^\perp} D \subset D$. Indeed, from $f = f_1 + f_1{}^\perp$, where $f \in D, f_1 \in D \cap H_1$ (by assumption), $f_1{}^\perp \in H_1{}^\perp$ and from linearity of D it follows $f_1{}^\perp \in D$. Let $h \in H_1 \cap D$ and $g \in H_1{}^\perp \cap D$. From the assumption that H_1 is an invariant subspace for A and A^* and from the elementary properties of the scalar product (,) we have

$$(Ag, h) = (g, A^*h) = 0 \tag{2}$$

It follows that Ag is orthogonal to $H_1 \cap D$. Using the density of D in H, continuity of the mapping P_{H_1} which maps H on the whole H_1, by standard arguments one can prove that $P_{H_1} D$ is dense in H_1 in the H_1 topology induced by the scalar product in H. From $P_{H_1} D \subset H_1 \cap D$ it follows that $H_1 \cap D$ is dense in H_1 also. Using continuity of the scalar product (Ag, h) with respect to $h \in H_1 \cap D$ we get $(Ag, h) = 0$ for any $h \in H_1$–i.e., $Ag \perp H_1$ or, $Ag \in H_1{}^\perp$ and the proof is completed. The proposition, together with Theorem 2, p. 132 of Akhieser & Clasman (1966), gives the following corollary: $P_{H_1} \in \mathscr{A}'$ if and only if H_1 is an invariant subspace for $\mathscr{A}$. (By $\mathscr{A}'$ we mean the set of all bounded operators B such that $BAf = ABf$, $B^*Af = AB^*f$, with $f \in D, A \in \mathscr{A}$.) This corollary generalises Powers' (1971) statement, Theorem 4.7, where the same correspondence between projections and invariant subspaces is proved for a special kind of operator algebras $\mathscr{A}$–"self-adjoint" algebras. For such algebras condition (1) is automatically fulfilled and the "weak" commutant used by Powers (1971) is identical with $\mathscr{A}'$ (see Powers, 1971, Lemma 4.6). In fact, Powers uses representations $\pi(\mathfrak{A}) \equiv \mathscr{A}, \pi(a) \equiv A$ of an abstract $*$ algebra $\mathfrak{A}$ $a \in \mathfrak{A}$ such that $\pi(a)^* \supset \pi(a^*)$ for any $a \in \mathfrak{A}$. But this difference is not essential for the validity of the proposition and the corollary. The only difference in the proofs appears in (2), which is modified to the form $(\pi(a)g, f) = (g, \pi(a^*)h)$ because of $\pi(a)^* = \pi(a^*)$ on D.

The above discussion leads to the conclusion that superselection rules in terms of unbounded observables are in one-to-one correspondence with the projections of the commutant $\mathscr{A}'$ as in the case of bounded observables, if the additional condition (1) if fulfilled, which from the physical point of view is rather natural.

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